Hadamard edf algorithm

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This algorithm computes the equivalent degrees of freedom

$$\operatorname{edf} V = \frac{2(EV)^2}{\operatorname{var} V}$$

for the unbiased fully overlapped estimator

$$V = \frac{1}{6(m\tau_0)^2(N-3m)} \sum_{n=1}^{N-3m} [\Delta^3_{m\tau_0} x(n\tau_0)]^2$$

of Hadamard variance $\sigma^2_H(m\tau_0)$ from N sampled time residuals $x(n\tau_0)$. In general, it is assumed that $x(t)$ is a process with stationary, Gaussian, *mean-zero* third differences; in particular, the computation is carried out for pure, unfiltered power-law noises $S_x(\nu) \propto \nu^\beta$, $\beta = -2,-3,-4,-5,-6$. We do not treat the situation $\beta \geq -1$ (flicker PM, white PM), where a high-frequency cutoff has to be taken into account.

Declarations

1. Given a noise model of white FM through random run FM for time residuals $x(t)$, take the (unscaled) generalized autocovariance function $R_x(t)$ and coefficients a_0, a_1 from the following table. For the flicker models, note that $R_x(0)$ must evaluate to 0.

Noise	β	μ	$R_x(t)$	a_0	a_1
WHFM	-2	-1	$-\lvert t\rvert$	7/9	1/2
FLFM	-3	0	$t^2 \ln\lvert t\rvert$	1.00	0.62
RWFM	-4	1	$\lvert t\rvert^3$	31/30	17/28
FWFM	-5	2	$-t^4 \ln\lvert t\rvert$	1.06	0.53
RRFM	-6	3	$-\lvert t\rvert^5$	1.30	0.54

2. Define the function

$$\begin{aligned} r(t) &= -\delta_1^6 R_x(t) \\ &= 20R_x(t) - 15R_x(t+1) - 15R_x(t-1) + 6R_x(t+2) + 6R_x(t-2) - R_x(t+3) - R_x(t-3). \end{aligned}$$

[This is the autocovariance function of the stationary process $\Delta_1^3 x(t)$.]

3. Define the function

$$\text{edfinv\_t}(m,M) = \frac{1}{Mr^2(0)}\left[r^2(0) + 2\sum_{j=1}^{\min(M,3m)} \left(1-\frac{j}{M}\right) r^2\left(\frac{j}{m}\right)\right], \quad \text{\# (edfinv-t)}$$

where m, M are positive integers. [This is a truncated version of the exact expression for 1/edf, in which the summation index goes to M (OK, $M-1$).]

4. Define the function

$$\text{edfinv\_c}(p) = \frac{1}{p}\left(a_0 - \frac{a_1}{p}\right)$$

for positive real p. [This is an upper bound for the limit of 1/edf as $m \to \infty$, $M/m \to p$, provided that

$p \geq 3$. The sum tends to an integral.]

5. Define a constant $J_{\max}$, the maximum number of terms that you will tolerate in (ref: edfinv-t). Suggest $J_{\max} = 100$.

Inputs

N = number of time residual samples
m = averaging time / sample period
Assume $3m < N$

Output

edf = equivalent degrees of freedom of fully overlapped Hadamard variance estimate

Procedure

$M = N - 3m$
$J = \min(M, 3m)$
$p = M/m$ [floating divide]
If $J \leq J_{\max}$ then
    [Small number of terms; compute the sum]
    edfinv = edfinv_t(m, M)
Elseif $M \geq 3m$ then
    [m large, $p \geq 3$, so use limiting form]
    edfinv = edfinv_c(p)
Else
    [M and m large, $p < 3$, so use sum with smaller proportional m, M]
    m' = nearest integer to $J_{\max}/p$
    edfinv = edfinv_t$(m', J_{\max})$
Endif
edf = 1/edfinv