CHAPTER 2

Synthesis of the Single Phase Network

At the time that the method of single sideband modulation described in Chapter 1 was conceived there were no known methods available for the design of sequence asymmetric polyphase networks. Consequently the following techniques were devised to meet the requirements as they arose.

1) Frequency transformations.
2) Image synthesis and transformation.
3) Insertion loss synthesis.
4) Passive R-C network design, including optimisation techniques.

The first three methods are discussed in this chapter while the design of passive R-C filters is dealt with separately in Chapter 4.

2.1 Frequency Transformations

Frequency transformations have long been known as a method of producing a new filter from an existing design. A high pass filter can be obtained for instance from a low pass design through the transformation:

$$\Omega_{HP} = {}^{-1}\!/W_{LP}$$

In this case a coil whose reactance is jWL is transformed to a capacitor whose reactance is $1/j\Omega(1/L)$. Similarly a capacitor

C is transformed to be a coil of value $^1/C$.

Now if this can be done for normal filters it seems reasonable to expect that other types of transformation are possible and particularly that asymmetric about zero characteristics can be obtained. Indeed this proves to be the case; a normal filter is symmetrical about zero frequency and simply shifting the characteristic sideways produces an asymmetric characteristic. The transformation required is

$$\Omega' = \Omega + \Delta \qquad \text{..... (1)}$$

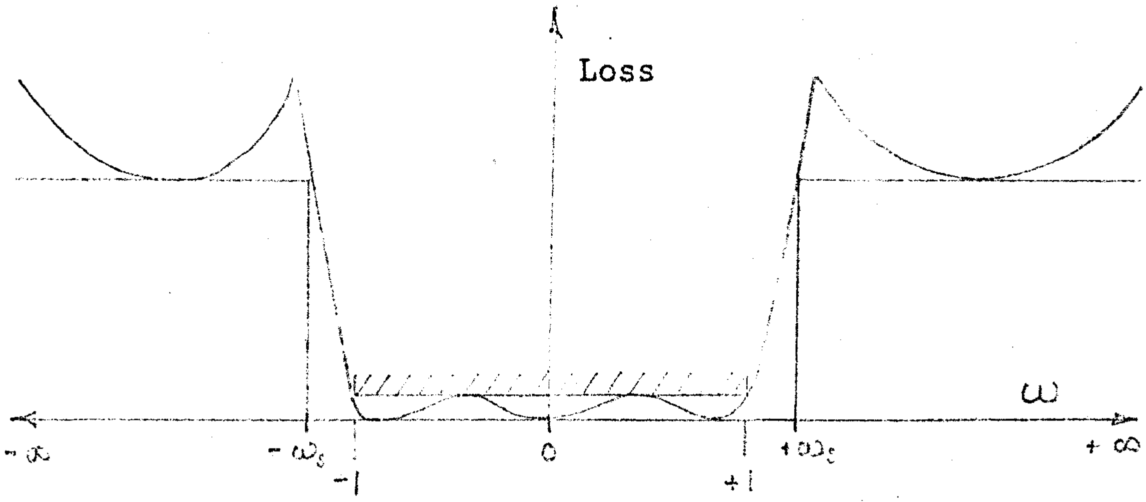
where Ω is the frequency scale of the original filter

Ω' is the frequency scale of the new asymmetric filter

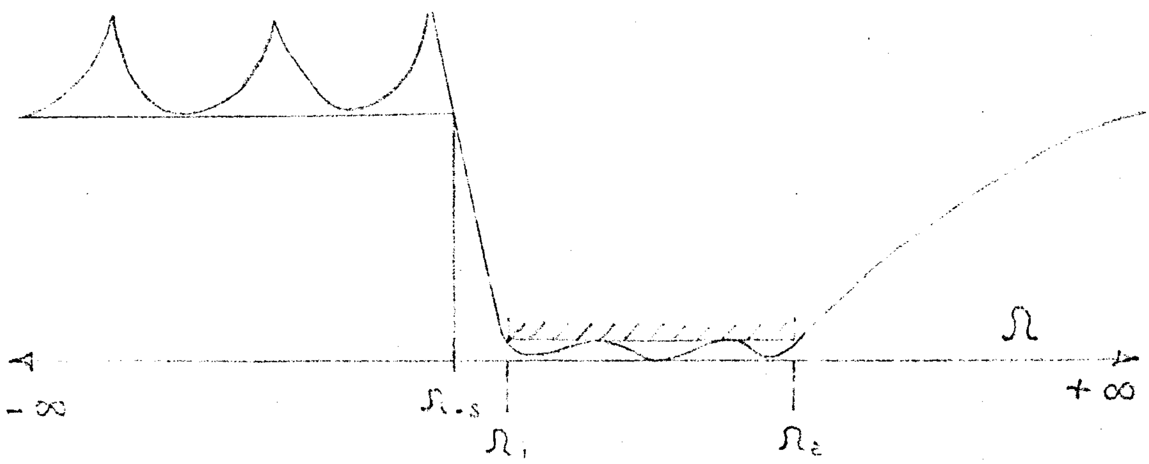
Δ is the frequency shift

If there was a coil L in the original filter its reactance would have been $j\Omega L$ and it would transform to $j(\Omega'-\Delta)L = j\Omega' L - j\Delta L$. Since Δ is fixed the coil transforms to a coil of the same value in series with a constant reactance. Similarly a capacitor C transforms to a capacitor of the same value in parallel with a constant admittance of $-j\Delta C$.

The fact that such a transformation produces unrealizeable elements is not unexpected since there is no physically detectable difference between positive and negative frequencies, and so the characteristic produced is meaningless in a physical single phase network. None-the-less the values of the "constant reactances" will be required when the complete polyphase network is designed.



Symmetrical Lowpass Filter



Transformed Filter

Figure 2.1.1 Frequency Transformation.

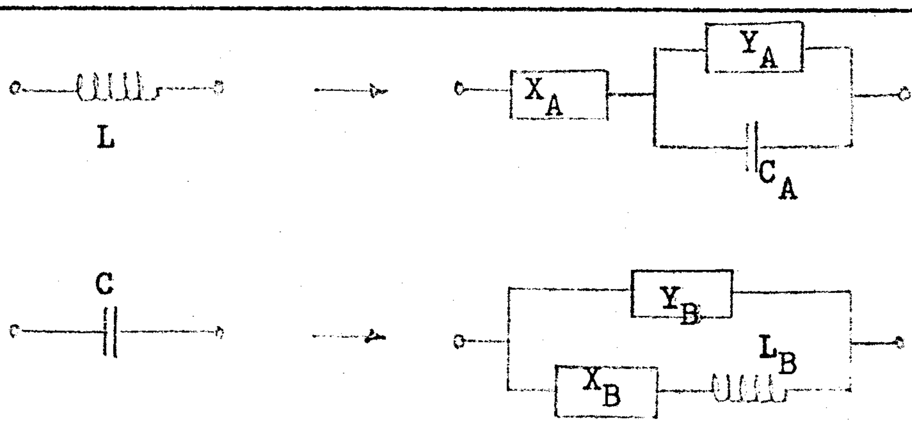


Figure 2.1.2 Element Transformations (equivalent to Fig 2.1.1 above)

There are many other more sophisticated frequency transforms that can be devised to meet particular requirements.

Consider for example

$$\Omega = \frac{(W+a)b}{W-W_2} \qquad \ldots\ldots (2)$$

This can be used to transform a symmetrical filter to one with arbitary cut off frequencies Ω_1 and Ω_2 as shown in Fig. 2.1.1. In addition it is arranged to shift the point $W = W_s$ to infinity on the Ω scale so that in the example shown all the peaks in both upper and lower stopbands of the original are shifted into the lower stopband of the new filter. In this case a and b are easily found by solving

$$\text{for } W = -1 \quad \Omega_1 = \frac{(-1+a)b}{-1-W_s}$$

$$\text{for } W = +1 \quad \Omega_2 = \frac{(1+a)b}{1-W_s} \qquad \ldots\ldots (3)$$

from which

$$a = \frac{W_s(\Omega_1+\Omega_2) + (\Omega_1-\Omega_2)}{W_s(\Omega_2-\Omega_1) - (\Omega_1+\Omega_2)} \qquad \ldots\ldots (4)$$

and

$$b = \frac{\Omega_1 \; (1+W_s)}{1-a} \qquad \ldots\ldots (5)$$

so that (substituting in (4))

$$\Omega = \frac{W\left[(\Omega_1+\Omega_2)-W_s(\Omega_2-\Omega_1)\right]-\left[W_s(\Omega_1+\Omega_2)+(\Omega_1-\Omega_2)\right]}{2\;(W-W_s)} \qquad \ldots\ldots (6)$$

As an example and to test the validity of the last transformation a third order filter was transformed such that $\Omega_1 = 0.25$ and $\Omega_2 = 3.4$. The original filter had the following specification:

Cut off frequency	$W = 1.00$
Passband ripple	$a_p = 0.20$ db
Stopband attenuation	$a_s = 33.6$ db
Stopband edge	$W_s = 2.50$

Using (4) and (5)

$$a = 1.4142012$$

$$b = -2.1125000$$

Hence in (6)

$$\bar{\Omega} = \frac{2.1125\ (W+1.4142012)}{2.5 - W}$$

or,

$$W = \frac{2.5\Omega-2.98750}{\Omega + 2.11250}$$

$$W = 2.5 - \frac{1}{0.1209373\Omega + 0.25547997}$$

The elements in the original filter are therefore transformed as follows:

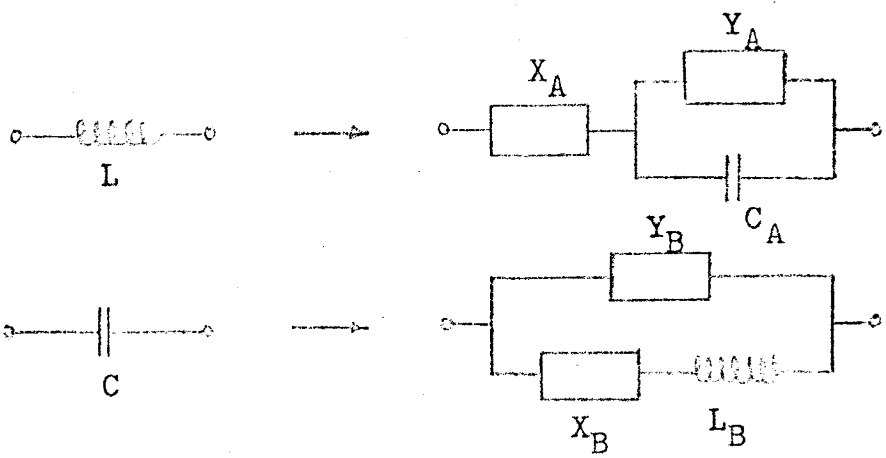
A coil:

$$jWL \rightarrow j2.5L + \frac{1}{j\ \frac{.1209373\Omega}{L} + j\ \frac{0.25547997}{L}}$$

A capacitor:

$$\frac{1}{jWC} \rightarrow \frac{1}{j2.5C + \dfrac{1}{j\,\dfrac{.1209373\Omega}{C} + j\,\dfrac{0.25547997}{C}}}$$

These transformations are illustrated in Fig. 2.1.3.



X_A = j2.5L Ohms $\qquad$ Y_B = j2.5C Mhos

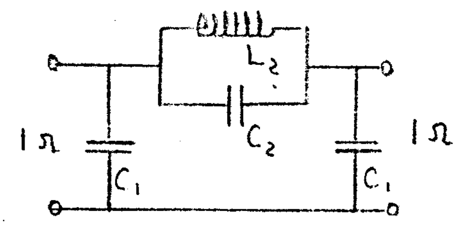
$Y_A = \dfrac{j0.255480}{L}$ Mhos $\qquad$ $X_B = \dfrac{j0.255480}{C}$ Ohms

$C_A = \dfrac{0.1209373}{L}$ Farads $\qquad$ $L_B = \dfrac{0.1209373}{C}$ H

Fig. 2.1.3 Impedance Transforms

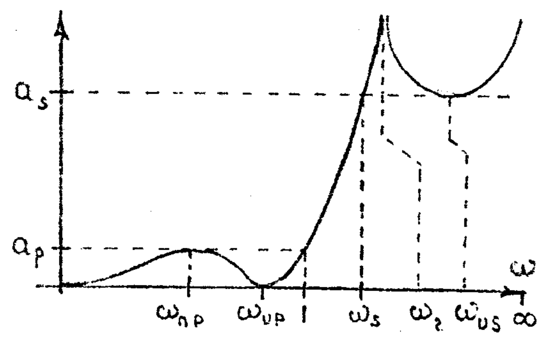
The original network, and the transformed network are shown in Fig. 2.1.4.

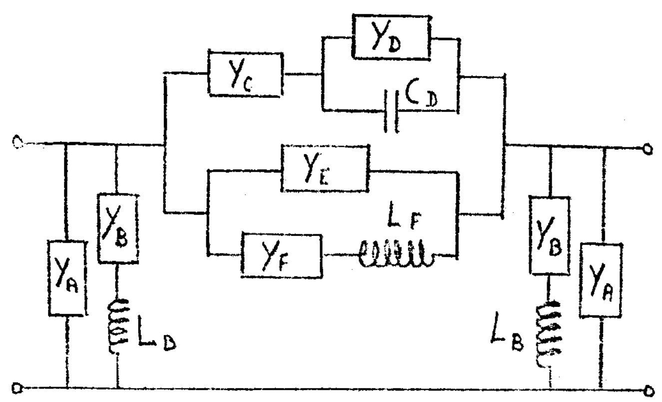
The transformed network was evaluated using a nodal analysis programme and the frequency response is shown in Fig. 2.1.5.



Original 3rd order filter

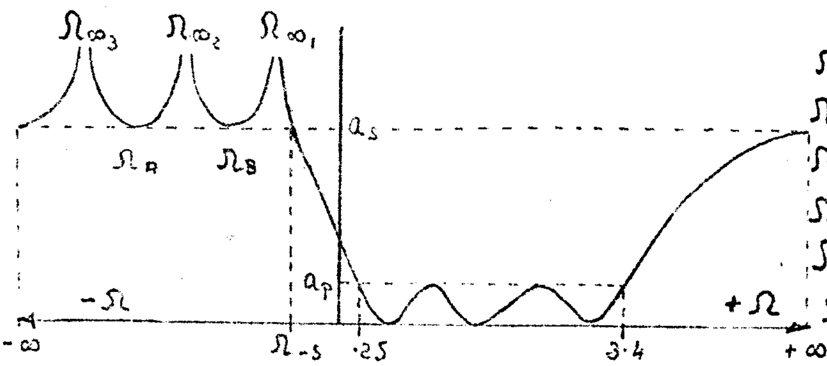
C_1	1.14134 F
C_2	0.11840 F
L_2	1.03524 H
ω_s	2.50000
ω_2	2.85631
ω_{up}	0.87526
ω_{np}	0.51634
ω_{us}	4.84176
a_p	0.2 dB
a_s	33.62 dB





Transformed filter

L_B	0.10596 H
L_F	1.02141 H
C_D	0.11682 F
Y_A	j2.85335
Y_B	-j4.46745
Y_C	-j0.38633
Y_D	j0.24678
Y_E	j2.96005
Y_F	-j0.46345



Ω_{-s}	-0.45900
$\Omega_{\infty1}$	-0.56876
$\Omega_{\infty2}$	-2.11250
$\Omega_{\infty3}$	-25.3194
Ω_A	-5.64350
Ω_B	-0.98623

Figure 2.1.4 Transformation of an elliptic low pass to an asymmetric band pass filter.

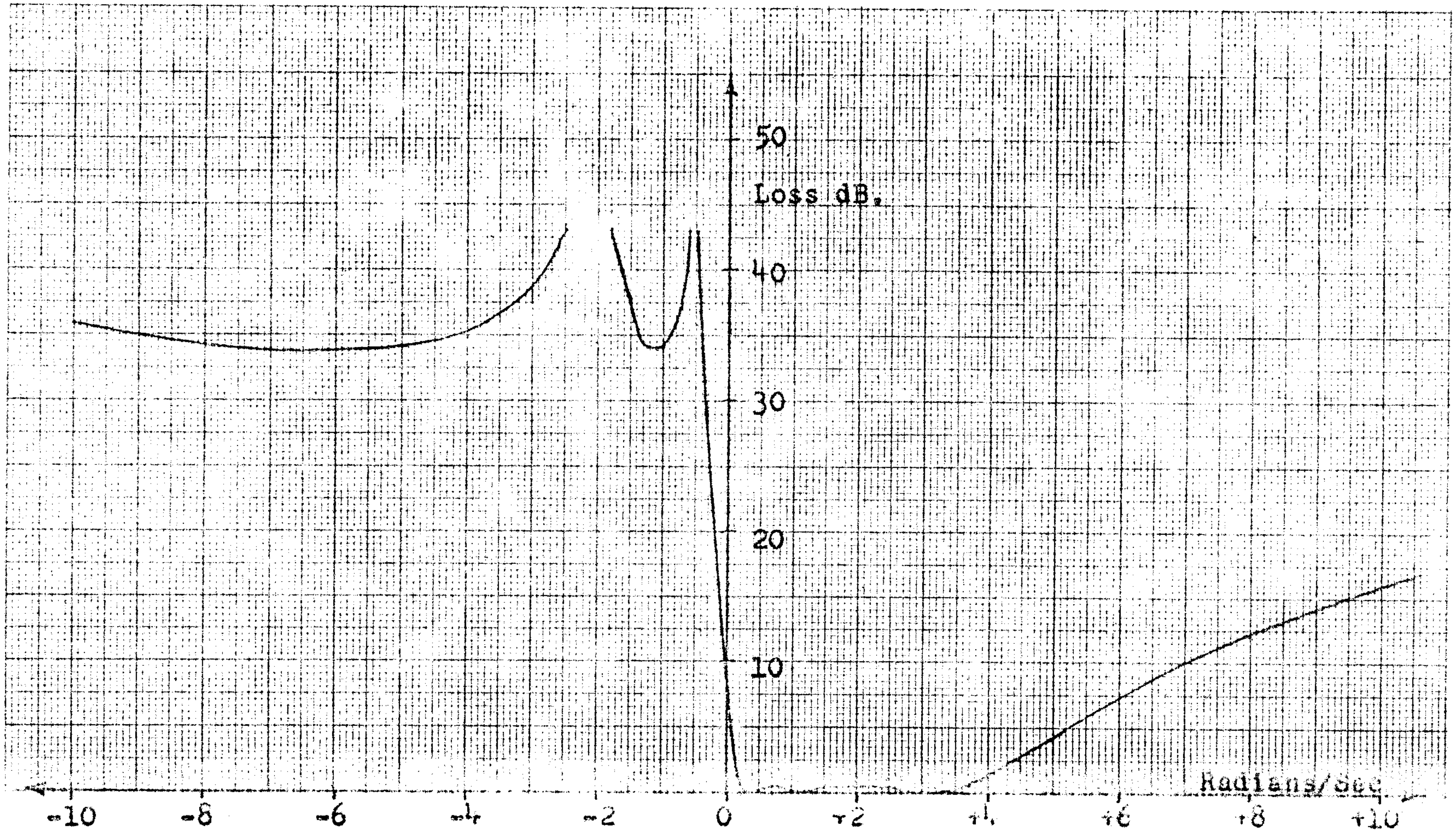


Figure 2.1.5 Frequency Response of transformed filter.

2.2 Synthesis using Image Methods

Image design methods are very important, not only because they enable rapid design but because they give a good insight into the types of structure possible with insertion loss design.

Two basic methods were considered:-

1) Direct synthesis of image sections from prescribed loss and image impedance functions.

2) By transformation from known image filter sections. Simultaneous transformation of the image impedance and frequency response allows the generation of a wide variety of sections.

2.2.1 Direct Image Synthesis

The lattice network forms the starting point and from this the equivalent ladder networks can be deduced using Bartletts' Bisection Theorem.

Fig. 2.2.1 shows the standard image filter relationships. Since the lattice arms are normally purely reactive it can be deduced that when they are of the same polarity a stopband exists and the image impedance is reactive. Similarly when they are of opposite polarity a passband exists and the image impedance is real. When the reactances of the two arms are equal then

$$q = \sqrt{Z_1/Z_2} = 1 \quad \text{and} \quad g_o = \text{infinite}$$

ie. a point of infinite attenuation exists.

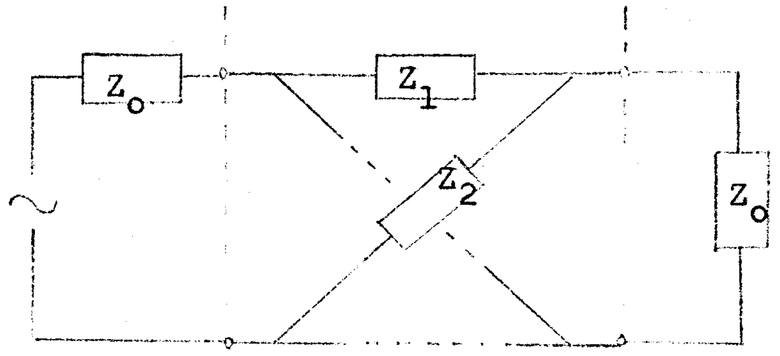


Image Section

Image Impedance $Z_o = \sqrt{Z_1 Z_2}$

Image Loss $G_o = A_o + jB_o = \mathrm{Log}_e\left[\frac{1+q}{1-q}\right]$ Nepers

-where $q = \sqrt{\frac{Z_1}{Z_2}}$

In the stopband q is real $A_o = 2\mathrm{Tanh}^{-1}(q)$ Np.

In the passband q imaginary $B_o = 2\mathrm{Tan}^{-1}(-jq)$ Rad.

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Synthesis

$$Z_1 = q.Z_o$$

$$Z_2 = Z_o/q$$

	Passband	Stopband
Z_o	real	imaginary
q	imaginary	real

Figure 2.2.1 Image Filter Relationships.

Using these considerations it is possible to synthesise sections with any desired complexity of stopbands and passbands. Figure 2.2.2 illustrates the very simplest filter with a passband from $-\infty$ to zero and stopband from zero frequency to ∞. A peak can be placed anywhere in the stopband. If the sign of the constant reactance X_2 is reversed the frequency response rotates about the zero frequency axis to reverse the positions of passband and stopband.

The image impedance of this section is not very practical being zero at infinite frequency and infinite at zero frequency and therefore difficult to terminate. It is, however, possible to synthesise more sophisticated image sections exercising independent control over the impedance and transfer functions (sometimes at the expense of network complexity) according to the following method.

Suppose that a filter is required with a passband from zero to some positive frequency ω_+. Let the stopband be from minus infinity to zero and from ω_+ to plus infinity with an attenuation peak in the upper (positive) stopband. Let the image admittance be zero at both cut-offs and one ohm maximum in the passband. These requirements, shown in Fig. 2.2.3, could be satisfied by the following relations:

$$Y_0 = \left[2 \sqrt{\omega(\omega_+ - \omega)}\right] / \omega_+ = {}^1/Z_0 = \sqrt{Y_1 Y_2}$$

$$q = \sqrt{\frac{Z_1}{Z_2}} = \sqrt{\frac{Y_2}{Y_1}} = \sqrt{\frac{\omega(\omega_\infty - \omega_+)}{(\omega - \omega_+)\omega_\infty}}$$

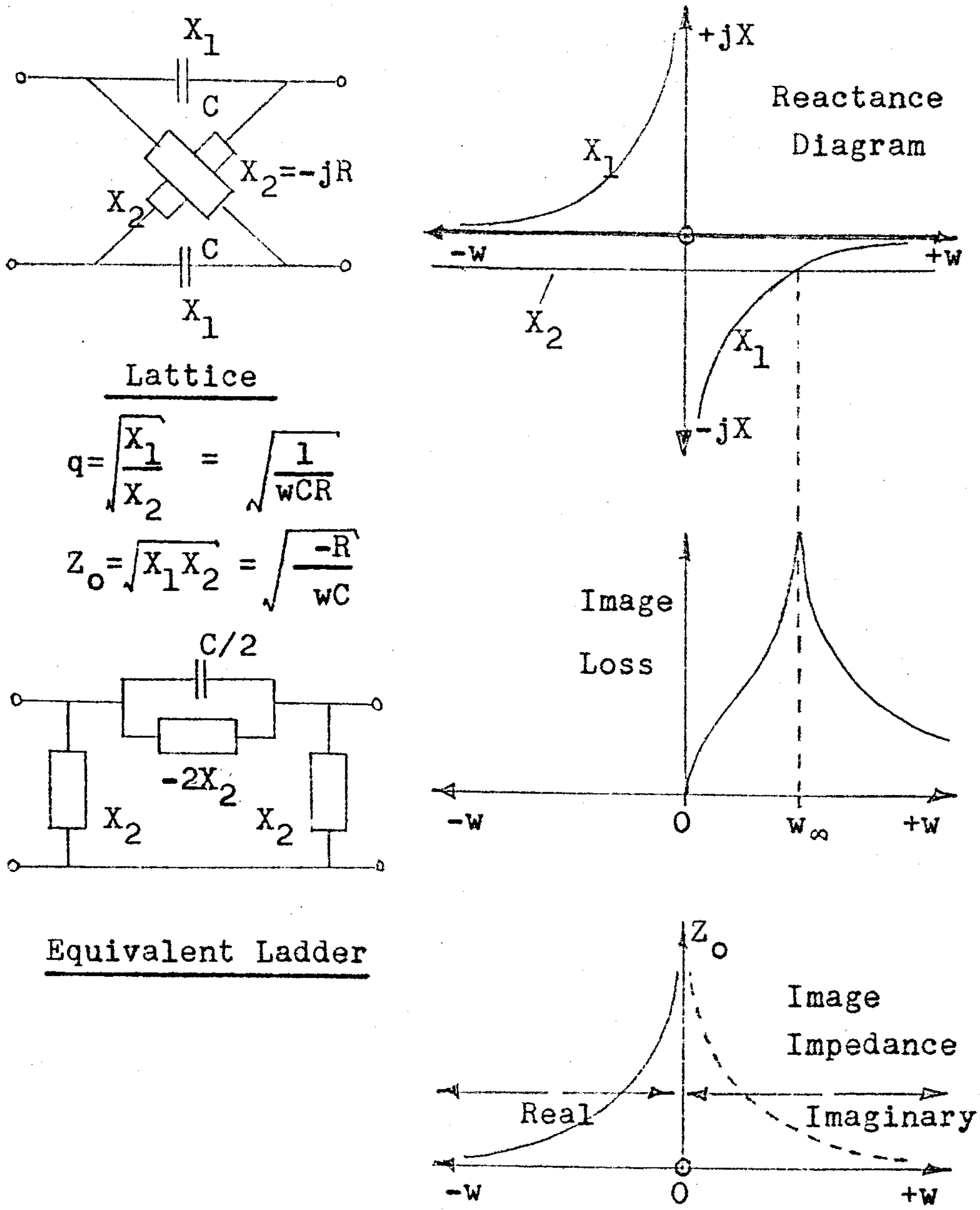


Figure 2.2.2 Simple Asymmetric Image Filter Section.

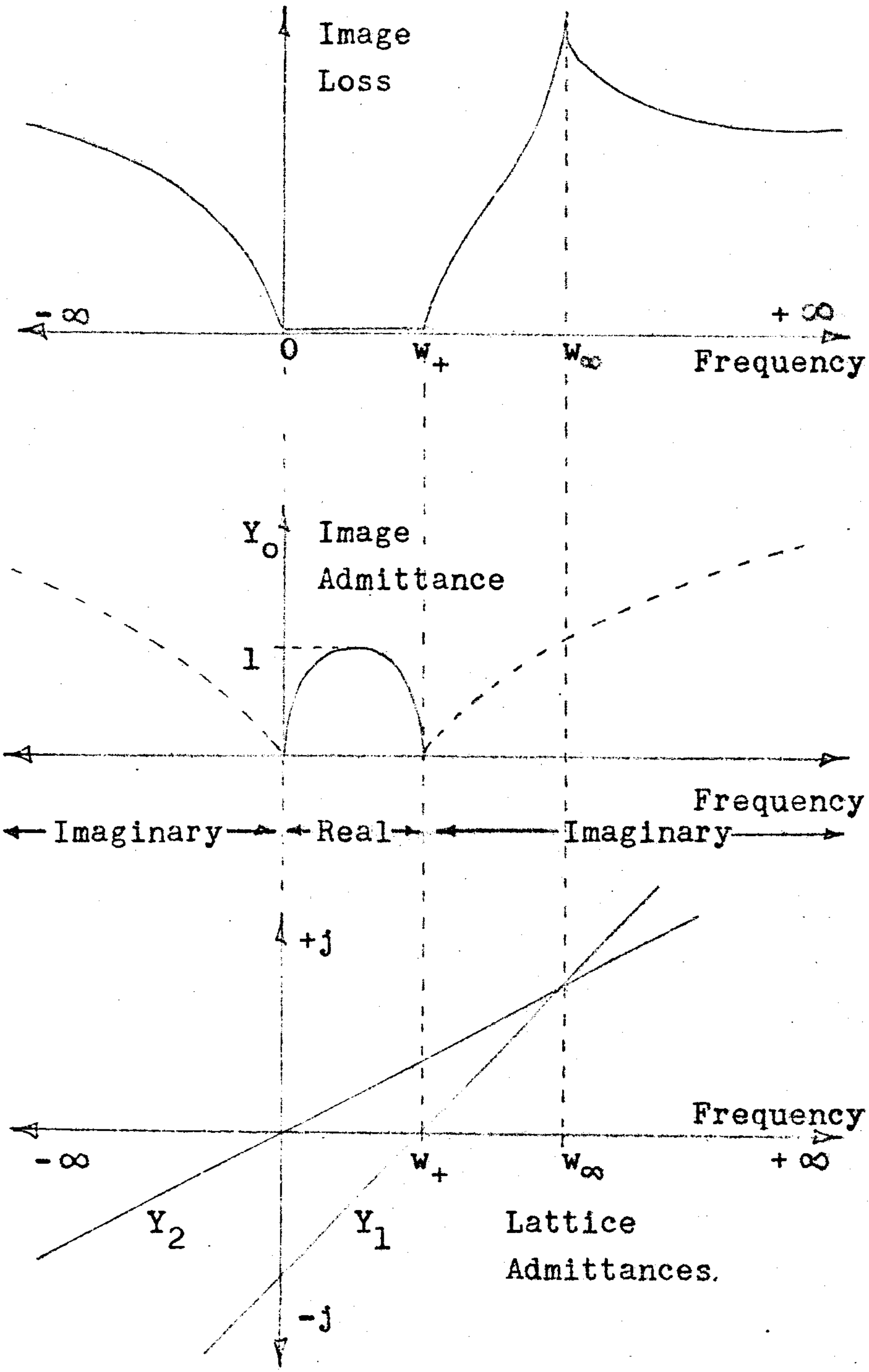


Figure 2.2.3 Image Filter Specification and resultant Lattice arm Admittances.

The expression for $Y_0^{\ 2}$ can be any rational function of ω as can that for q^2. The expressions given are the simplest that satisfy the pole and zero requirements of the specification. The factor q must be arranged to be zero or infinity at the cut-off frequencies and unity at points where infinite attenuation is required since image loss $g_o = \log_e \left(\frac{1 + q}{1 - q}\right)$ Nepers

Having defined Y_0 and q the lattice admittances can be determined:

$$Y_1 = Y_0/q = 2j\left(\frac{\omega}{\omega_+} - 1\right) \sqrt{\frac{\omega_\infty}{\omega_\infty - \omega_+}}$$

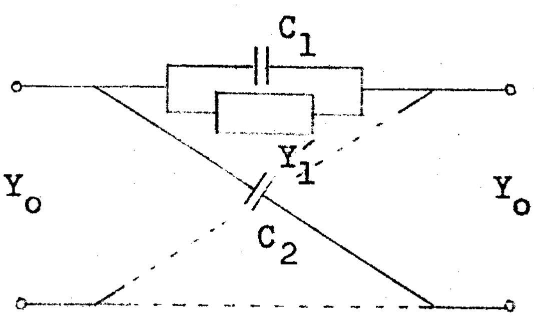
$$Y_2 = qY_0 = 2j\,\frac{\omega}{\omega_+} \sqrt{\frac{\omega_\infty - \omega_+}{\omega_\infty}}$$

Then by one of the standard methods the expressions for Y (or Z) can be broken into factors giving the individual circuit elements. In this case Y_1 is a capacitor in parallel with a constant reactance Y_1' and Y_2 is a capacitor C_2. The resultant lattice and its equivalent ladder (derived by Bartletts' Bisection Theorem) are depicted in Fig. 2.2.4.

Using the foregoing method a variety of image sections can be synthesised giving flexibility in design.

2.2.2 Transformation of Existing Image Sections

Because so many types of conventional image filter sections already exist it is useful to be able to transform them to asymmetric about zero sections. This is particularly true where for example, the final filter structure is

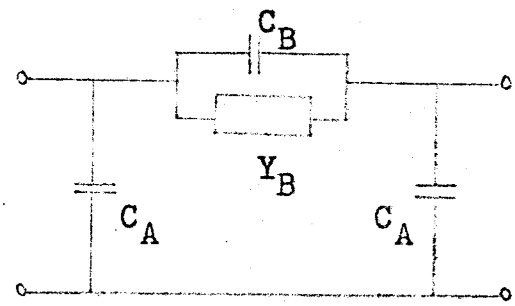


Lattice

$$C_1 = \frac{2}{\omega_+}\sqrt{\frac{\omega_\infty}{\omega_\infty - \omega_+}}$$

$$Y_1' = -j2\sqrt{\frac{\omega_\infty}{\omega_\infty - \omega_+}}$$

$$C_2 = \frac{2}{\omega_+}\sqrt{\frac{\omega_\infty - \omega_+}{\omega_\infty}}$$



Ladder Equivalent

$$C_A = C_2$$

$$C_B = (C_1 - C_2)/2$$

$$Y_B = Y_1'/2$$

Figure 2.2.4 Networks matching the specifications of Fig. 2.2.3

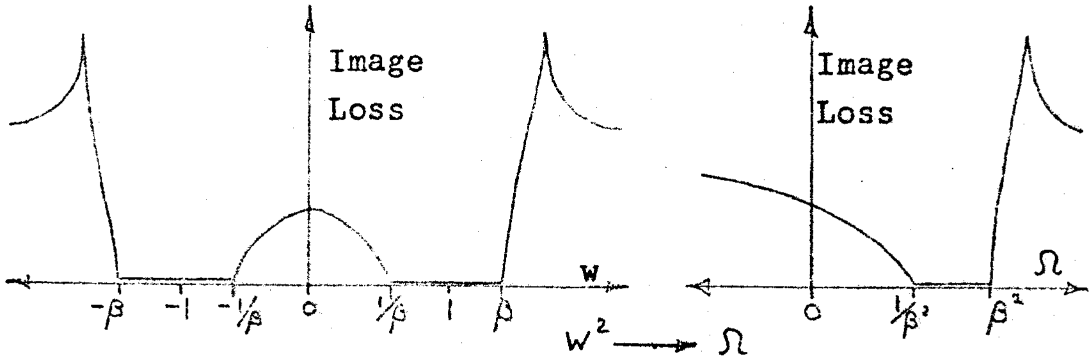
required to be of a particular form for some practical reason and by choosing known conventional sections as the starting point this can be achieved.

Although the methods described in section 2.1 apply there is the further possibility with image filters of making a simultaneous image impedance transformation.

In Fig. 2.2.5 the transformation is shown of an asymmetric bandpass filter in ω to an asymmetric about zero filter in Ω through the simple transform

$$\omega^2 \longrightarrow \Omega$$

However, applying this to the elements of the original filter section would result in irrational impedances in Ω. For example a coil whose reactance was $j\omega L$ would become a reactance $j\sqrt{\Omega}L$. This can be avoided by premultiplying all reactances by ω (alternatively $1/\omega$) thus ensuring that they are functions in ω^2 and hence in Ω after transformation. The image impedance is also modified by the same factors and so the criteria for choosing the terminating resistors must be changed accordingly. Further transformations are still possible using the methods of section 2.1 to move or stretch the frequency scale to get the desired final effect. A case of interest for single sideband modulation where this can be applied with advantage is when the passband is required to extend from zero to $\omega = +1$. Setting the lower cut-off to zero results, in some cases, in a reduction in the number of elements required. In fig. 2.2.5 where the original filter



Asymmetric Band Pass Filter ⟶ Asymmetric about Zero Filter

Image Admittance $Y_o(w) \longrightarrow w.Y_o(w)$

Capacitor $Y = jwC \longrightarrow jw^2C \longrightarrow j\Omega C$

Coil $Y = \dfrac{1}{jwL} \qquad \dfrac{1}{jL}$ Constant Admittance

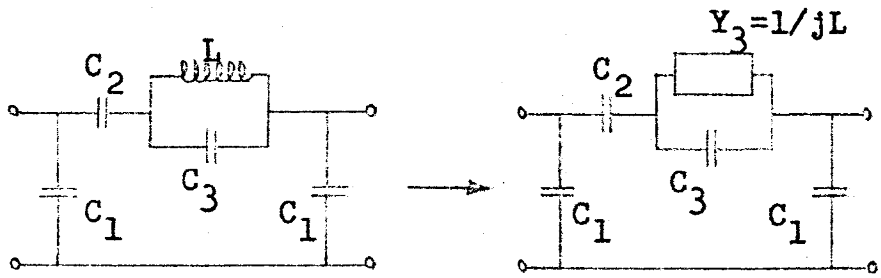


Figure 2.2.5

Transformation of Existing Image Sections to Asymmetric about Zero Sections had cut-off frequencies at $\omega = \beta$ and $\omega = {}^{1}/_{\beta}$ the transformation required is a frequency scaling to make the pass-band 1 radian wide ie. to make $\beta^2 - \frac{1}{\beta^2} = 1$ and a linear shift of $\frac{1}{\beta^2}$ to place the lower cut-off at zero ie.

$$\Omega' = \Omega - \frac{1}{\beta^2}$$

with a corresponding transformation of element values and image impedances.

2.3 Insertion Loss Design

The constraints of conventional network synthesis require that the modulus square of the transfer function being symmetrical about zero frequency, must be the ratio of two polynomials in even powers of ω. For assymmetric-about-zero functions, however, odd powers of ω must exist in the expression for the modulus. In consequence, the actual transfer function must contain some imaginary terms in even powers of ω and real terms in odd powers of ω. Mathematically H(p), the transfer function, can be expressed as

$$H(p) = \frac{\sum_r (A_r + jB_r)p^r}{\sum_r (C_r + jD_r)p^r} \quad \text{where } p = j\omega \qquad \ldots\ldots (1)$$

2.3.1 Double Terminated Loss-Less Network Considerations

Consider a loss-less two port network. If one port is terminated in a one ohm resistor then the impedance seen looking into the other port is the driving point impedance Z_D.

$$\text{Let } Z_D = \frac{M_1 + n_1}{M_2 + n_2} \qquad \text{..... (2)}$$

where

M_1 and M_2 are of the form $\sum_r \{A_r p^{2r} + jB_r p^{2r+1}\}$ ie. real

and n_1 and n_2 are of the form $\sum_r \{jA_r p^{2r} + B_r p^{2r+1}\}$ ie. imaginary

$$\text{Now } Z_D = \frac{\Delta_Z + Z_{11}}{Z_{22} + 1} \qquad \text{..... (3)}$$

where Z_{11} and Z_{22} and Δ_Z are defined by the impedance matrix

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \qquad \text{..... (4)}$$

Δ_Z being $Z_{11}\ Z_{22} - Z_{12}^2$

V_1, I_1 being the conditions at port 1

V_2 and I_2 being the conditions at port 2

This is illustrated in Fig. 2.3.1.

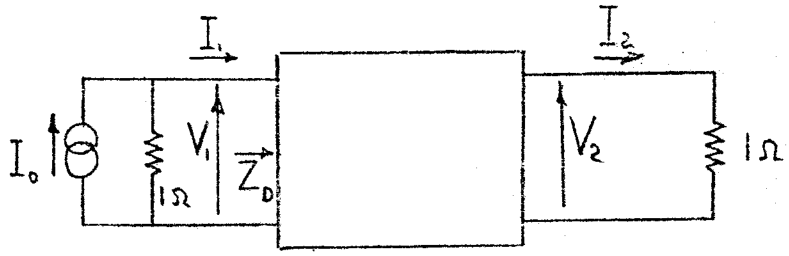


Fig. 2.3.1 2 Port Relationships

Z_D may also be written as

$$Z_D = Z_{11} \frac{1 + \frac{1}{Y_{22}}}{1 + Z_{22}} \qquad \text{..... (5)}$$

$$\text{where } Y_{22} = Z_{11}/\Delta_Z \qquad \ldots\ldots (6)$$

Since the elements defining the matrix must be purely reactive to make the network lossless there are two cases to consider.

<u>Case A</u>

$$Z_D = Z_{11}\,\frac{1 + \frac{1}{Y_{22}}}{1 + Z_{22}} = \frac{M_1}{n_2}\left[\frac{\frac{n_1}{M_1} + 1}{\frac{M_2}{n_2} + 1}\right] \qquad \ldots\ldots (7)$$

ie. $Z_{11} = \dfrac{M_1}{n_2}$, $Z_{22} = \dfrac{M_2}{n_2}$, $Y_{22} = \dfrac{M_1}{n_1}$

Hence $\Delta_Z = \dfrac{n_1}{n_2}$ and

$$Z_{12} = \frac{\sqrt{M_1 M_2 - n_1 n_2}}{n_2} \qquad \ldots\ldots (8)$$

<u>Case B</u>

$$Z_D = Z_{11}\,\frac{1 + \frac{1}{Y_{22}}}{1 + Z_{22}} = \frac{n_1}{M_2}\left[\frac{\frac{M_1}{n_1} + 1}{\frac{n_2}{M_2} + 1}\right] \qquad \ldots\ldots (9)$$

ie. $Z_{11} = \dfrac{n_1}{M_2}$, $Z_{22} = \dfrac{n_2}{M_2}$, $Y_{22} = \dfrac{n_1}{M_1}$

Hence $\Delta_Z = \dfrac{M_1}{M_2}$ and

$$Z_{12} = \frac{\sqrt{n_1 n_2 - M_1 M_2}}{n_2} \qquad \ldots\ldots (10)$$

Note that if the network is reversed

$$Z_{2D} = \frac{M_2 + n_1}{M_1 + n_2} \qquad \text{(Case A)} \qquad \ldots\ldots (11)$$

and if $M_1 = M_2$ $\quad Z_{1D} = Z_{2D}$ therefore Symmetrical network

or if $n_1 = n_2$ $\quad Z_{1D} = {}^1/Z_{2D}$ therefore Antimetrical network

or

$$Z_{2D} = \frac{M_1 + n_2}{M_2 + n_1} \qquad \text{(Case B)} \qquad \ldots\ldots (12)$$

and if $M_1 = M_2$ $\quad Z_{1D} = {}^1/Z_{2D}$ therefore Antimetrical network

or if $n_1 = n_2$ $\quad Z_{1D} = Z_{2D}$ therefore Symmetrical network

Consider again Fig. 2.3.1. The transfer ration $\frac{E_2}{I_0}$ is given by

$$\frac{E_2}{I_0} = \frac{Z_{12}}{(1+Z_D)\ (1-Z_{22})} = \frac{\sqrt{M_1 M_2 - n_1 n_2}}{M_1 + M_2 + n_1 + n_2} \qquad \ldots\ldots (13)$$

therefore

$$\left|\frac{E_2}{I_0}\right|^2 = \left[\frac{E_2}{I_0}\right] \times \left[\frac{E_2}{I_0}\right]$$

$$= \frac{\sqrt{M_1 M_2 - n_1 n_2}}{M_1 + M_2 + n_1 + n_2} \cdot \frac{\sqrt{M_1 M_2 - (-n_1).(-n_2)}}{M_1 + M_2 - n_1 - n_2}$$

$$= \frac{M_1 M_2 - n_1 n_2}{(M_2 + M_2)^2 - (n_1 + n_2)^2} \qquad \ldots\ldots (14)$$

$$= \tfrac{1}{4} \left[1 - \left| \frac{1 - Z_D}{1 + Z_D} \right|^2 \right] \qquad \ldots\ldots (15)$$

therefore $\left|T(p)\right|^2 = 1 - \left|R(p)\right|^2$ $\qquad \ldots\ldots (16)$

where $$T(p) = \frac{2\,E_2(p)}{I_0(p)} \qquad \text{..... (17)}$$

$$R(p) = \frac{1 - Z_D(p)}{1 + Z_D(p)} \qquad \text{..... (18)}$$

or $$\left|H(p)\right|^2 = 1 + \left|K(p)\right|^2 \qquad \text{..... (19)}$$

where $$H(p) = \frac{I_0(p)}{2E_2(p)} = \frac{(M_1 + M_2) + (n_1 + n_2)}{M} \qquad \text{..... (20)}$$

$$K(p) = \frac{(M_1 - M_2) + (n_1 - n_2)}{M} \qquad \text{..... (21)}$$

$$M\bar{M} \text{ being } 4(M_1\,M_2 - n_1\,n_2) \qquad \text{..... (22)}$$

Note that in the foregoing analysis it was necessary to avoid the usual assumption that $\left|H(p)\right|^2 = H(p)\,.\,H(-p)$. This is only true for the case where H(p) is symmetrical on the $p = j\omega$ axis. In the assymetric - about-zero case $\left|H(\omega)\right| \neq \left|H(-\omega)\right|$ by definition.

Given a suitable transfer function, in the form of H(p) or T(p) it is now possible to deduce a corresponding K(p) or R(p) and from that the expression for the driving point impedance Z_D. After that it is a matter of logical decomposition of Z_D to synthesise the network.

2.3.2 Synthesis of Equiripple Passband Arbitary Stopband Functions

The method used here has been adapted from work by Orchard, Bingham and others (refs. E1-E3) to allow the synthesis of functions of the form shown in Fig. 2.3.2a. The filter has an equiripple passband from ω_L to ω_U and a stopband of arbitary specification at all other frequencies.

By making the frequency transformation

$$Z^2 = \frac{\omega - \omega_U}{\omega - \omega_L} \qquad \text{..... (23)}$$

the passband becomes imaginary in Z while the whole stopband transforms into the whole real Z domain as in Fig. 2.3.2b. This transformation allows the arbitary design of the stop-band to be made using template methods as will be shown later.

Let the attenuation poles occur at $Z^2 = m_\sigma^2$ where $\sigma = 1,2,3....r$ and form the polynomial

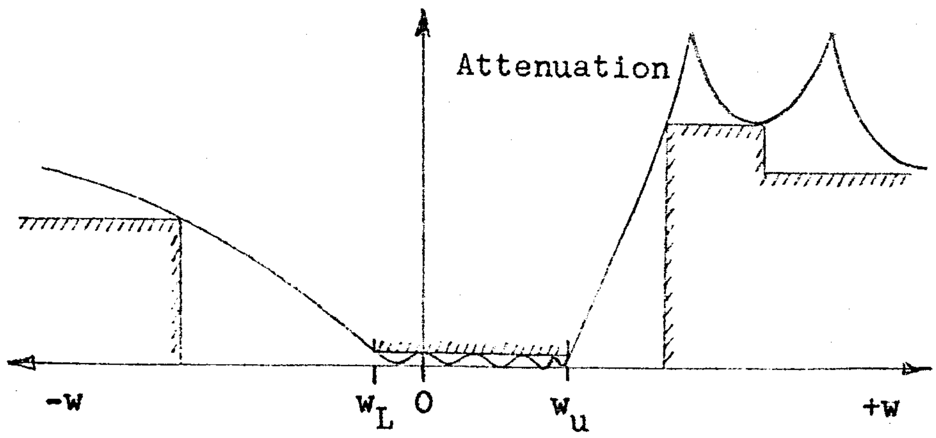
$$E + Z\,F = \prod_{\sigma=1}^{r} (Z + m_\sigma)^2 \qquad \text{..... (24)}$$

where E and F are even polynomials in Z.

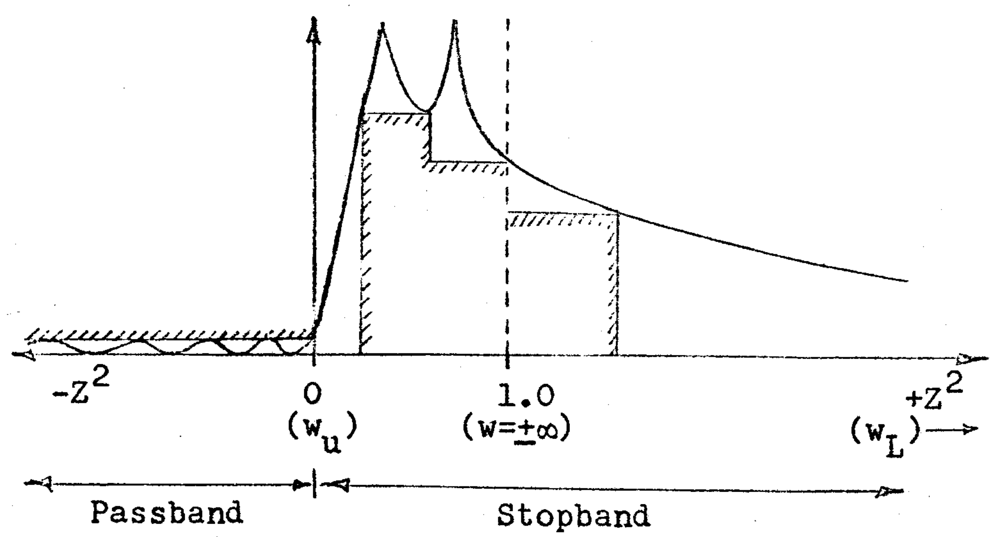
The insertion transfer function H(p) of section 2.3.1 can now be defined in such a way as to give an equiripple passband by setting

$$H(p)\,.\,\overline{H(p)} = 1 + \frac{tE^2}{E^2 - Z^2F^2} \qquad \text{..... (25)}$$

$$= 1 + \frac{t}{4}\left[\frac{E + ZF}{E - ZF} + 2 + \frac{E - ZF}{E + ZF}\right]$$



A) Typical Filter Specification



B) Transformed Specification

Figure 2.3.2 Transformation to the Z^2 frequency domain.

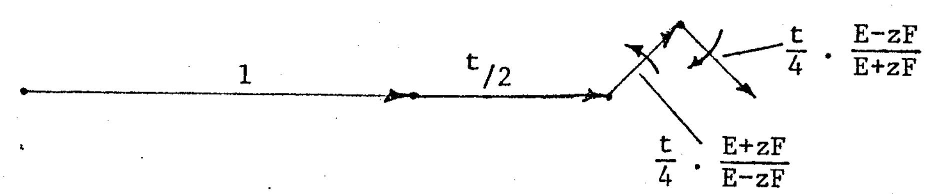


Fig. 2.3.2-C Equiripple nature of Equation (25) in passband when Z is imaginary.

Figure 2.3.2-C illustrates (25) graphically where, since Z is imaginary in the passband $\frac{E+ZF}{E-ZF}$ is of constant modulus. The vector sum is real, oscillating between 1 and 1+t giving a passband ripple of $10 \log_{10} (1+t)$dB.

Now the stopband loss is given by $A_s = 10 \log_{10} |H(p)|^2$ dB. If this loss is sufficiently large (>10db) then A_s may be approximated by

$$A_s = 10 \log_{10} \left[\frac{t}{4} \cdot \frac{E + ZF}{E - ZF}\right] \text{dB.} \qquad \ldots\ldots (26)$$

$$\text{or } A_s = 10 \log_{10} \frac{t}{4} + \sum_{\sigma=1}^{\sigma=r} \left\{10 \log_{10} \frac{1 + Z/m_\sigma}{1 - Z/m_\sigma}\right\} \ldots\ldots (27)$$

$$\text{Now if } Z/m_\sigma = e^{\gamma-\mu} \qquad \ldots\ldots (28)$$

where $Z = e^{\gamma}$ and $m_\sigma = e^{\mu_\sigma}$ then

$$A_s = 10 \log_{10} \frac{t}{4} + \sum\left\{10 \log_{10} \operatorname{Coth}\left(\frac{\gamma-\mu_\sigma}{2}\right)\right\} \qquad \ldots\ldots (29)$$

Therefore, by plotting the stopband requirements on the γ frequency scale where $\gamma = \log_e Z = \frac{1}{2} \log_e \left(\frac{\omega-\omega_U}{\omega-\omega_L}\right)$ and then using identical templates of the form $10 \log_{10} \operatorname{Coth} \frac{\gamma}{2}$ displaced appropriately along the frequency axis the values of μ_σ and hence m_σ can be determined. Due allowance must be made for the passband ripple factor $10 \log_{10} \frac{t}{4}$.

Factorization of the Transfer Function

Having determined the positions of the attenuation poles on the Z frequency axis it is desirable to be able to obtain the transfer function H(p) in factored form. It has been shown (refs. E1-E3) that this can be done most accurately in the Z plane.

$$\text{Now } H(p).\overline{H(p)} = 1 + \frac{tE^2}{E^2 - Z^2F^2} \qquad \ldots\ldots (30)$$

$$= \frac{(E\sqrt{1+t} + ZF)\ (E\sqrt{1+t} - ZF)}{(E + ZF)\ (E - ZF)}$$

$$\text{therefore } H(p) = \frac{E\sqrt{1+t} + ZF}{E + ZF} \qquad \ldots\ldots (31)$$

and this can be factored using numerical methods to

$$H(p) = \frac{\prod (Z^2 + mZ + n)}{E + ZF} \qquad \ldots\ldots (32)$$

Consider a single quadratic factor of the numerator $Q = Z^2 + mZ + n$. Having found this it can be transformed back into a factor in p according to the following method.

$$\text{Now } Q.\bar{Q} = (Z^2 + mZ + n)\ (Z^2 - mZ + n)$$

$$= Z^4 + (2n - m^2)Z^2 + n^2$$

$$= \left(\frac{\omega-\omega_U}{\omega-\omega_L}\right)^2 + (2n - m^2)\left(\frac{\omega-\omega_U}{\omega-\omega_L}\right) + n^2$$

$$= \frac{R\,\omega^2 - S\omega + T}{(\omega - \omega_L)^2} \qquad \ldots\ldots (33)$$

where $R = (1 + n)^2 - m^2$

$S = 2(1 + n)\ (\omega_U + n\omega_L) - m^2\ (\omega_U + \omega_L)$

$T = (\omega_U + n\omega_L)^2 - m^2\ \omega_U\omega_L$

Therefore, taking the roots of (33) which give zeros in the left hand half of the p plane

$$H(p) = \frac{A+pB}{M} = \frac{\prod \sqrt{R}\left[P - j\,\frac{S}{2R} + \frac{1}{2}\sqrt{\frac{4T}{R} - \frac{S^2}{R^2}}\right]}{M\prod(\omega-\omega_L)} \ldots\ldots (34)$$

Now $M = E + ZF$ ..... (35)

therefore $M\bar{M} = E^2 - Z^2F^2 = \prod (m_\sigma + Z)^2 (m_\sigma - Z)^2$

$$= \prod \left(m_\sigma^{\ 2} - \frac{\omega-\omega_U}{\omega-\omega_L} \right)^2 \qquad \text{..... (36)}$$

And finally from (34) and (36)

$$H(p) = \prod_{\sigma=1}^{\sigma=r} \left\{ \frac{\sqrt{R}\left[P - j\frac{S}{2R} + \frac{1}{2}\sqrt{\frac{4T}{R} - \frac{S^2}{R^2}} \right]}{\omega(m_\sigma^{\ 2} - 1) + (\omega_U - \omega_L)} \right\} \qquad \text{..... (37)}$$

Derivation of K(p)

From (17) $|K(p)|^2 = |H(p)|^2 - 1$

$$= \frac{tE^2}{E^2 - Z^2F^2}$$

$$= \frac{\sqrt{t}\ E}{E + ZF} \cdot \frac{\sqrt{t}\ E}{E - ZF}$$

thus $K(p) = \dfrac{\sqrt{t}\ E}{E + ZF}$ ..... (38)

Now E is an even polynomial in Z and therefore of the form $\sum C_\sigma\ Z^{2\sigma}$ and can be transformed to a polynomial in $j\omega$ through the substitution $Z^2 = \dfrac{\omega-\omega_U}{\omega\ \omega_L}$.

The denominator E + ZF can be transformed as described for H(p)

Therefore:

$$K(p) = \frac{t\sum C_\sigma \left(\frac{\omega-\omega_U}{\omega-\omega_L} \right)^\sigma}{\prod \left(m_\sigma^{\ 2} - \frac{\omega-\omega_U}{\omega-\omega_L} \right)} \qquad \text{..... (39)}$$

Design Example

A computer programme was written to design filters automatically and accurately. The quadratic factors of H(p) were generated approximately and then improved first using Lins' method and secondly Bairstow's method.

However, to test not only the computer programme but also to check for flaws in the theory, a simple filter was designed by hand so that all the intermediate results were available.

Specification

Passband:

Upper cut off ω_U = 1.0

Lower cut off ω_L = 0.0

Attenuation ripple = 0.1 dB

Stopband:

Peak at $\omega = -0.25$

Peak at $\omega = \infty$

Passband ripple = 0.1 = $10 \log_{10} (1 + t)$

therefore t = 0.0233

$$\text{Now } E + ZF = \prod_{\sigma=1}^{\sigma=2} (Z + m_\sigma)^2 \qquad \text{(from (24))}$$

$$m_\sigma = \sqrt{\frac{\omega_\infty - \omega_U}{\omega_\infty - \omega_L}} \qquad \text{(from (23))}$$

$$\text{therefore} \quad m_1 = \sqrt{\frac{-0.25 - 1.00}{-0.25 - 0.00}} = 2.236068$$

$$m_2 = \sqrt{\frac{1 - {1.00}/{\infty}}{1 - {0.00}/{\infty}}} = 1.0$$

therefore $E + ZF = (Z + 2.236068)^2 . (Z + 1)^2$

therefore $E = Z^4 + 14.9442719\ Z^2 + 5$

$F = 6.47213596\ Z^2 + 14.47213596$

Now $\sqrt{1 + t} = 1.0115829$

therefore to evaluate H(p) using (31) $E\sqrt{1+t} + ZF =$

$1.0115829\ Z^4 + 15.11737018\ Z^2 + 5.05791459$

$+ 6.47213596\ Z^3 + 14.47213596\ Z$

$= \sqrt{1+t}\left[Z^4 + a_3 Z^3 + a_2 Z^2 + a_1 Z + a_o\right]$

where $a_3 = 6.398028125$

$a_2 = 14.944271900$

$a_1 = 14.306425800$

$a_o = 5.00$

In this simple case the quadratic factors of the quartic were evaluated from the explicit quartic solution which states that the quadratic factors are given by:

$$Z^2 + \left[\frac{a_3}{2} \mp \left(\frac{a_3^{\ 2}}{4} + U_1 - a_2\right)^{\frac{1}{2}}\right] Z + \left[\frac{U_1}{2} \mp \left(\frac{U_1^{\ 2}}{4} - a_o\right)^{\frac{1}{2}}\right]$$

where U_1 is the real root of the cubic

$U^3 - a_2 U^2 + (a_1 a_3 - 4a_o)U - (a_1^{\ 2} + a_o a_3^{\ 2} - 4a_o a_2) = 0$

which may be solved by Cardan's method. As a result $U_1 = 6.85570227$ and the two quadratic factors are

$Z^2 + 4.673330805\ Z + 6.058967702$

and $Z^2 + 1.7246873205\ Z + 0.825223082$

The quadratic factors in Z can now be transformed to factors in p.

	1st factor	2nd factor
m	4.673330805	1.7246973205
n	6.058967702	0.825223082
R	27.898004205	0.356858450
S	- 7.722085409	0.675865317
T	1.0	1.0
α	0.137948551	- 0.946965545
β	0.129222724	1.380394063

The factors in p are given by

$$p + j\alpha + \beta$$

The numerator of H(p) therefore becomes

$$\sqrt{R_1 R_2}\ (p + j\beta_1 + \alpha_1)\ (p + j\beta_2 + \alpha_2) \times 1.0115829$$

$$= 3.1970041156\ p^2 - j2.586430644\ p + 0.987908831$$

$$+ 4.826243710\ p + j0.217568598$$

$$= m_1 + m_2 + n_1 + n_2$$

The denominator of H(p) is from (37):

$$\prod \left[\omega(m_\sigma{}^2 - 1) + (\omega_U - \omega_L)\right]$$

$$= (4\omega + 1) = M$$

Now $K(p) = \dfrac{\sqrt{tE}}{E+ZF} = \dfrac{m_1 - m_2 + n_1 - n_2}{M}$

and the numerator becomes

$$\sqrt{tE} = 0.1526432559 \left[Z^4 + 14.9442719\ Z^2 + 5\right]$$

$$= 0.1526432559 \left[(\omega-1)^2 + 14.9442719\omega(\omega-1) + 5\omega^2\right] / \omega^2$$

And therefore

$$m_1 - m_2 + n_1 - n_2 = -3.197001855\ p^2 + j2.58642883p$$

$$+ 0.1526432559$$

Since $m_1 - m_2 + n_1 - n_2$ is real then $n_1 = n_2$ from (2).

Synthesis of the Network

From the evaluation of H(p) and K(p) we have

$$m_1 + m_2 = 3.1970041156p^2 - j2.586430644p + 0.987908831$$

$$m_1 - m_2 = -3.197001855p^2 + j2.58642883p + 0.1526432559$$

$$2n_1 = 4.82625108p + j0.217568594 = 2n_2$$

therefore

$$Z_D = \frac{m_1 + n_1}{m_2 + n_2}$$

$$= \frac{4.82625108p + 1.14055209 + j0.2175685984}{6.39400597p^2 + 4.82625108p - j5.17285947p + 0.835265575 + j0.2175685984}$$

This represents the input impedance of the network when terminated by a resistor at its far end. In order to break the network into its component parts some assumptions must be made about its structure. Examining Z_D it can be seen that as $p \rightarrow \infty$

$$Z_D \rightarrow {}^1/(1.32483906p)$$

which means that there must be a shunt capacitor of value 1.32483906 at the input. Removing this capacitor C_1 the admittance left is

$$Y_D' = \frac{3.315203115p - j5.46110286p + .835265575 + j.2175685984}{4.82625108p + 1.14055209 + j0.2175685984}$$

Now there is a transmission zero at $\omega = -0.25$ and this must be realized either with a series or with a shunt resonator circuit. Assuming a shunt resonator which produces a short circuit across the ladder network at $\omega = -0.25$.

$$Y_D' = -j0.535908705. \quad (\text{at } \omega = -0.25)$$

Therefore there must be a series branch consisting of a constant reactance $Y_1 = {}^1/Y_D' = j1.86598468$ immediately preceding the shunt resonator.

Removing the series constant reactance Y_D' reduces to

$$Y_D'' = \frac{3.315203115p - j5.46110286p + 0.835265575 + j0.2175685984}{-(p+j0.25)\ (5.364109339 + j6.186134096)}$$

The denominator has been factorized so that Y_D'' can be split into partial fractions:

$$Y_D'' = \frac{p\ Y_2}{p + {Y_2}/{C_2}} \quad + \quad R(p) \quad \text{and as a result}$$

$$Y_2 = j0.39522695 \quad C_2 = 1.58090779$$

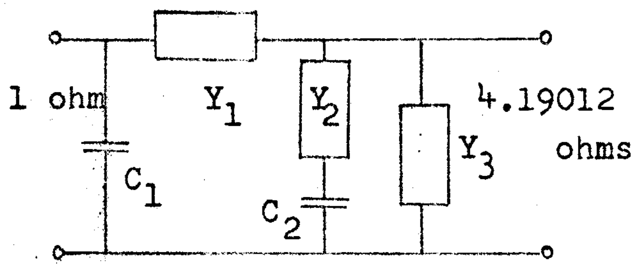
and $\quad R(p) = 0.23865667 + j0.3476253$

$$= ({}^1/R_T) + Y_3.$$

The network is therefore completed by a shunt resonator consisting of Y_2 and C_2 in series, a shunt admittance Y_3 and the terminating resistor R_T. The final network is shown in Fig. 2.3.3.

Because of the way the elements were extracted the terminating resistor came out to be 4.19011964 ohms instead of 1 ohm. If different assumptions had been made about the structure, the value of R_T would also have been different. Using Norton's theorem transformations are possible which would allow the termination to be made 1Ω.

To check the final result the transducer loss was computed using a nodal analysis programme and is plotted in



Y_1 = -j0.5359087 mho
Y_2 = +j0.3952269 "
Y_3 = +j0.3476253 "
C_1 = 1.3248391 F
C_2 = 1.5809078 F

Figure 2.3.3 Final Circuit of Insertion Loss designed Filter.

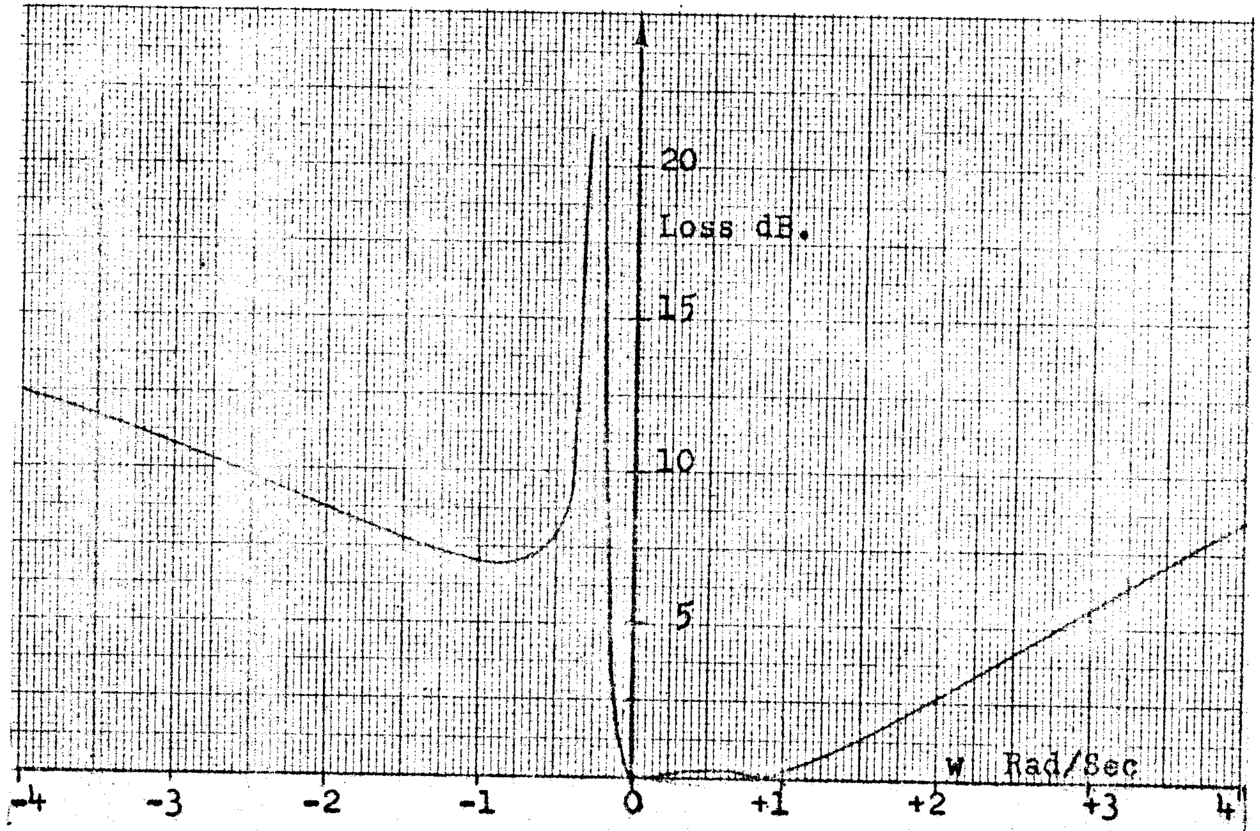


Figure 2.3.4 Frequency Response Of Insertion Loss designed Filter.

Fig. 2.3.4. (Transducer loss being the power in the load with the network in circuit minus the power in the load with the network replaced by an ideal matching transformer).

2.3.3 Elliptic Function Filters

A common requirement, particularly in single sideband modulation, is for filters which provide an equiripple passband and an equal minima stopband. Such filters are sometimes called 'Elliptic' because of the use of Jacobian elliptic functions in the solution of the approximation problem. They are also sometimes called 'Cauer' filters after Cauer who was one of the first to describe them (E4). Considerable work has been done on the computation and tabulation of such filters (E8) and these are sometimes in a suitable form for frequency transformation using the methods of section 2.1. However, in many cases the wanted attenuation characteristics in both the passband and the stopband are not available. Also, the tables are usually in the form where, to meet the realizeability criteria of LC networks, they have already been subjected to frequency transformations. Accordingly, if an asymmetric about zero filter is required with special characteristics in the transfer function, it may be extremely difficult to find the right set of data in the tables. A particular case of interest is a class of R-C passive (resistor/capacitor) networks where the criterion is that the attenuation zeros of the transfer function are required to be purely imaginary. This imposes special constraints on the relationships between the passband and stopband attenuations and the passband-stopband frequency transition ratio. Such functions are unlikely to be found in any published work and it therefore becomes necessary to consider direct synthesis from the specification.

Consider the characteristic of Fig. 2.3.5 which has a passband from $\omega = -{}^{1}/x$ to $\omega = -x$ with ripple a_p and a stopband from $\omega = +{}^{1}/x$ to $+ x$ with an attenuation of a_s minimum. Most other characteristics could be reduced to this by frequency transformation. It is this particular one however, which is most interesting for applications in single sideband modulation. The transfer function can be expressed as

$$|H(p)|^2 = 1 + t^2|K(p)|^2 \qquad \ldots\ldots (40)$$

where $|K(p)|$ is an auxilliary function which oscillates between $+\ \varepsilon$ and $-\ \varepsilon$ over the passband and $+\ {}^{1}/\varepsilon$ through infinity to $-\ {}^{1}/\varepsilon$ over the stopband. Solving for H(p) and K(p) in this form facilitates network synthesis according to the method of section 2.3.1.

The passband and stopband attenuations are given by:

$$a_p = 10 \log_{10} (1 + t^2\varepsilon^2) \quad \text{dB} \qquad \ldots\ldots (41)$$

and

$$a_s = 10 \log_{10} (1 + t^2/\varepsilon^2) \quad \text{dB.} \qquad \ldots\ldots (42)$$

<u>The Approximation Problem</u>

A function F = K(p) is required as shown in Fig. 2.3.5 which oscillates about zero over the passband and just touches the lines $\pm\,\varepsilon$. Over the stopband it must oscillate about infinity and just touch the $\pm\ {}^{1}/\varepsilon$ lines.

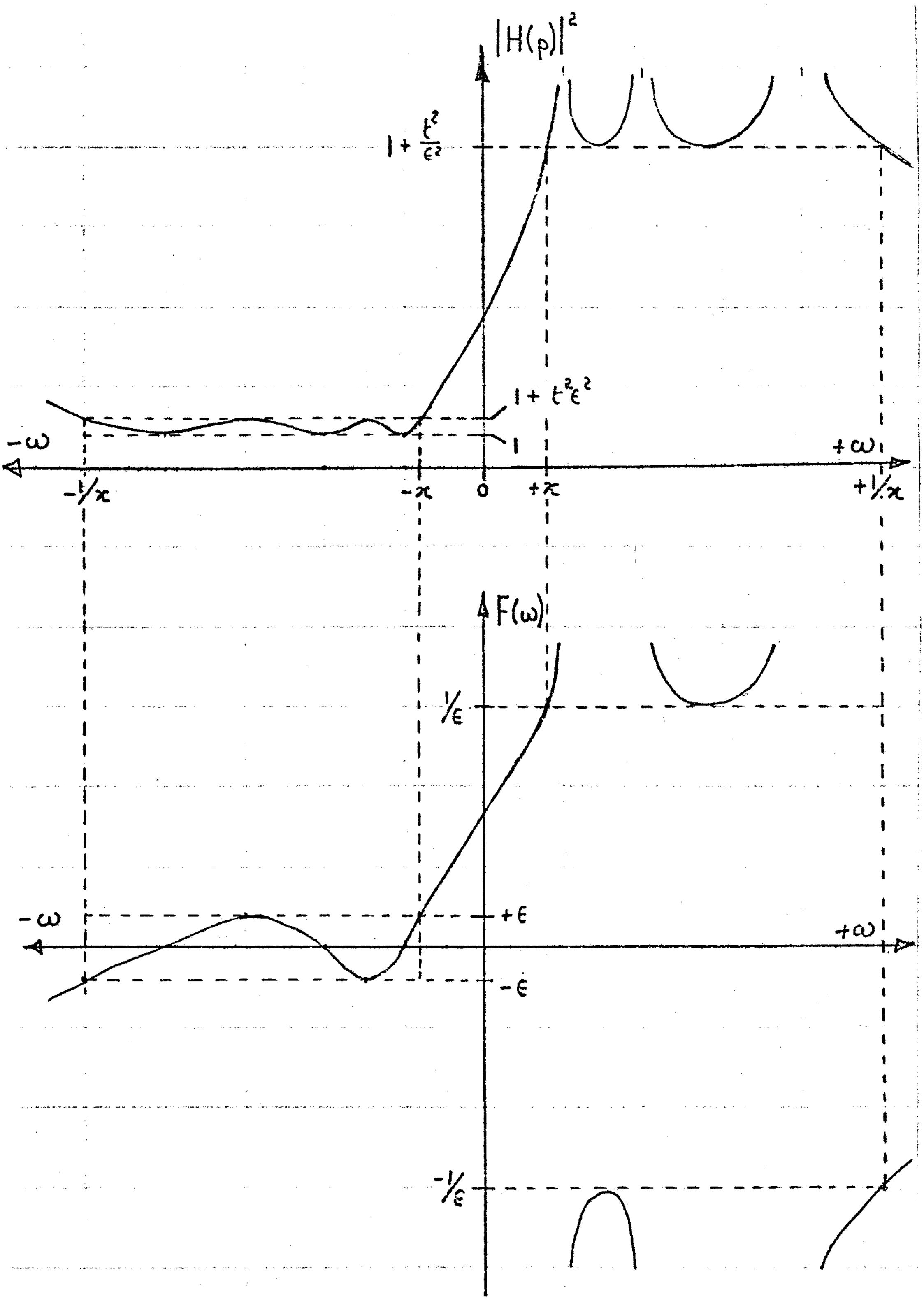


Figure 2.3.5 Relation between F(w) and H(w) for the equiripple passband / equiminima stopband filter.

Considering the roots of the following expressions

$$\left.\begin{aligned} \varepsilon^2 - F^2 &= 0 \\ 1 - \varepsilon^2 F^2 &= 0 \end{aligned}\right\} \qquad \ldots\ldots (43)$$

$$\left.\begin{aligned} \omega^2 - x^2 &= 0 \\ 1 - \omega^2 x^2 &= 0 \\ \left(\frac{dF}{d\omega}\right)^2 &= 0 \end{aligned}\right\} \qquad \ldots\ldots (44)$$

it can be seen that expressions (43) have together exactly the same number of roots and at the same frequencies as have the expressions (44) taken together.

The following equality may therefore be stated:

$$(\varepsilon^2 - F^2)\ (1 - \varepsilon^2 F^2) = C^2 (\omega^2 - x^2)\ (1 - \omega^2 x^2) \left(\frac{dF}{d\omega}\right)^2$$

$$\text{or} \quad \frac{d\omega}{\sqrt{(\omega^2 - x^2)\ (1 - x^2\omega^2)}} = \frac{C\ dF}{\sqrt{(\varepsilon^2 - F^2)\ (1 - \varepsilon^2 F^2)}}$$

$$= dU \qquad \ldots\ldots (45)$$

where C is a constant and U is an intermediate variable.

Integrating between equivalent passband limits:

$$U = \int_{-1/x}^{\omega'} \frac{d\omega}{\sqrt{(\omega^2 - x^2)\ (1 - x^2\omega^2)}} \qquad \ldots\ldots (46)$$

$$= C \int_{-\varepsilon}^{F(\omega')} \frac{dF}{\sqrt{(\varepsilon^2 - F^2)\ (1 - \varepsilon^2 F^2)}} \qquad \ldots\ldots (47)$$

Integrating (46) and (47)

$$U = dn^{-1}\left[-\omega\, x,\ \sqrt{1 - x^4}\right] \qquad \text{..... (48)}$$

$$= C\left\{K_1 + sn^{-1}\left[\frac{F(\omega')}{\varepsilon},\ \varepsilon^2\right]\right\} \qquad \text{..... (49)}$$

where dn(x,y) and sn(x,y) are two of the doubly periodic Jacobian Elliptic Functions. K_1 is the complete elliptic integral of the first kind, modulus ε^2.

Rearranging (48):

$$\omega = -\frac{1}{x}\, dn\left[U_1,\ \sqrt{1 - x^4}\right] \qquad \text{..... (50)}$$

which is effectively a translation formula between the ω or p plane and a new 'U' frequency plane. Considering (50) and using the standard relationships for elliptic functions the following table can be drawn relating U and ω.

U	ω	
0	$-{}^1/x$	Passband
$K/2$	-1	
K	$-x$	
$K+jK'$	0	Transition
$K+2jK'$	$+x$	Stopband
$K/2+2jK'$	$+1$	
$+2jK'$	$+{}^1/x$	
$+jK'$	∞	Transition through infinity
0	$-{}^1/x$	

Going from top to bottom the table covers the whole real frequency axis while traversing a rectangle in the U plane.

Note that K is the complete elliptic integral of modulus $m = \sqrt{1 - x^4}$ whereas K′ is the complete elliptic integral of modulus $m_1 = \sqrt{1 - m^2} = x^2$.

Considering now (49) this can be written as

$$F(\omega) = \varepsilon.\text{sn}\left[\frac{U}{C} - K_1, \ \varepsilon^2\right] \qquad \ldots\ldots (51)$$

The constant C allows one degree of freedom in deciding the order of the function F(ω). C must therefore be arranged so that as U passes from 0 to K then $\left(\frac{U}{C} - K_1\right)$ passes through an appropriate number of multiples of K_1.

$$\text{ie.} \quad \left(\frac{K}{C} - K_1\right) - \left(\frac{0}{C} - K_1\right) = 2n\ K_1$$

$$\text{or} \quad C = \frac{K}{2n\ K_1} \quad \text{and therefore}$$

$$F(\omega) = \varepsilon.\text{sn}\left[\frac{(2n\ U - K)K_1}{K}, \ \varepsilon^2\right] \qquad \ldots\ldots (52)$$

Figure 2.3.6 shows how the function F(ω) oscillates over the passband for the example of Figure 2.3.5.

It can also be observed that the zeros of F(ω) occur at:

$$U_Z = \left(\frac{2r - 1}{2n}\right)K \qquad r = 1,2..n \qquad \ldots\ldots (53)$$

Also the maxima occur at

$$U_m = \frac{r}{n}K \qquad r = 1,2..n \qquad \ldots\ldots (54)$$

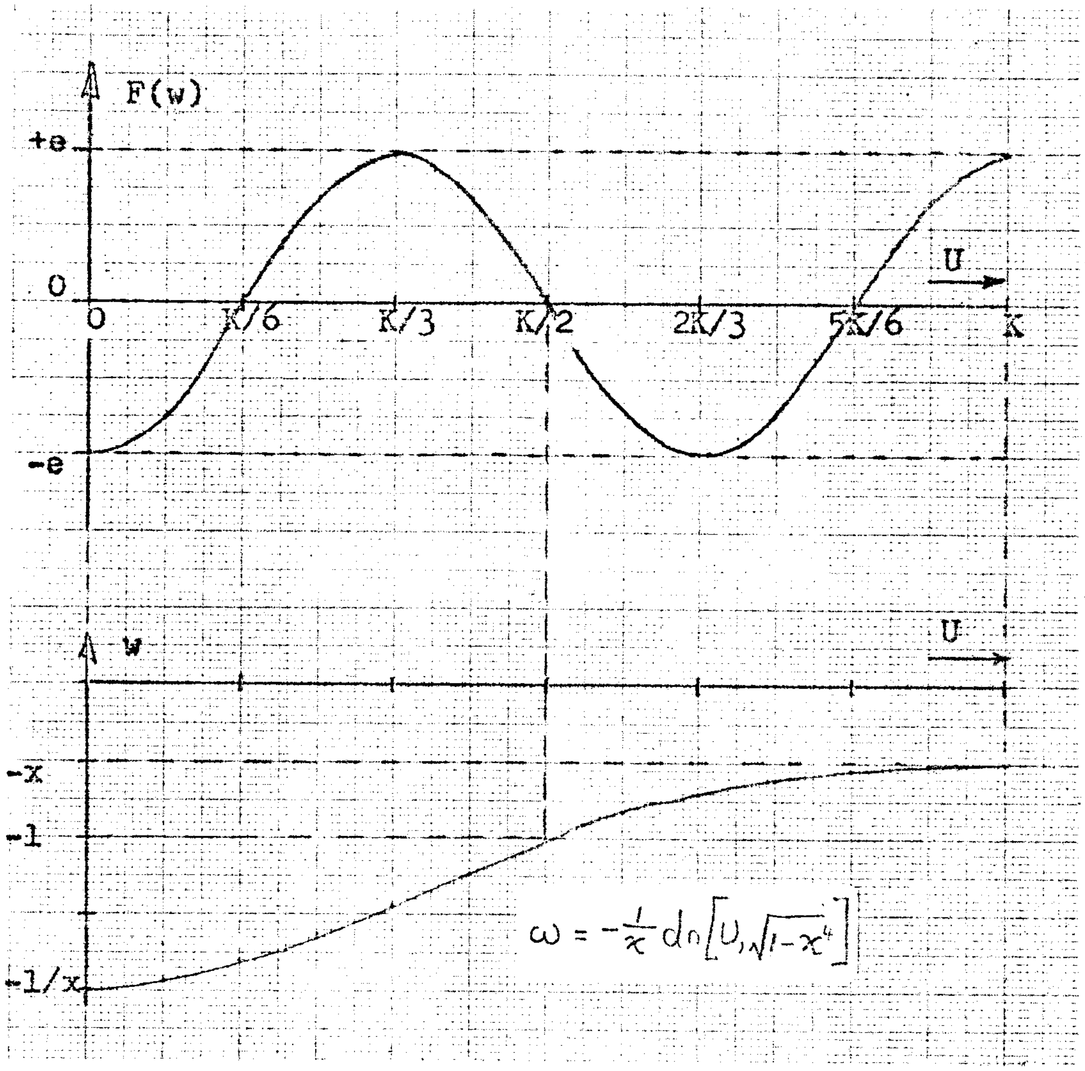


Figure 2.3.6 Showing the oscillation of F(w) over the passband and the relation between U and w.
Example shows case for N=3 x=.5

The Transfer Function

Now from (40) and (52)

$$|H(p)|^2 = 1 + t^2 |K(p)|^2$$
$$= 1 + t^2 \varepsilon^2 sn^2 \left[\left(2n \frac{U-K}{K} \right) K_1, \; \varepsilon^2 \right] \qquad \ldots\ldots (55)$$

but $|H(p)|^2 = H(p) \,.\, \overline{H(p)}$

$$\text{therefore } H(p) = 1 - jt\varepsilon\, sn \left[\left(2n \frac{U-K}{K} \right) K_1, \; \varepsilon^2 \right] \qquad \ldots\ldots (56)$$

The zeros of H(p) occur therefore when

$$sn \left[\left(2n \frac{U-K}{K} \right) K_1, \; \varepsilon^2 \right] = {}^{j}/_{t\varepsilon}$$

$$\text{Let } U = jV + {}^{K}/_{2n} \qquad \ldots\ldots (57)$$

$$\text{then } sn \left[\frac{jV.2n.K_1}{K}, \; \varepsilon^2 \right] = {}^{j}/_{t\varepsilon}$$

$$\text{or } sc \left[\frac{V.2n.K_1}{K}, \; \sqrt{1-\varepsilon^4} \right] = {}^{1}/_{t\varepsilon}$$

$$\text{or at } V_0 = \frac{K}{2nK_1} sc^{-1} \left[{}^{1}/_{t\varepsilon}, \; \sqrt{1-\varepsilon^4} \right] \qquad \ldots\ldots (58)$$

Since sc is periodic roots also occur at $V = V_0 + j\frac{sK}{n}$

$$\text{and therefore at } U = jV_0 - \left(\frac{1+2s}{2n} \right) K \qquad \ldots\ldots (59)$$

where $S = 0,1,2 \ldots.n$

The zeros of H(p) in the p plane are therefore at

$$\underline{\omega_Z = -\frac{1}{x} \, dn \left[jV_0 - \left(\frac{1+2s}{2n} \right) K, \; \sqrt{1-x^4} \right]} \qquad \ldots\ldots (60)$$

where K is the complete elliptic integral modulus $\sqrt{1-x^4}$ and V_0 is defined by (58).

The poles of H(p) in the p plane are at the same frequencies as those of K(p) and hence F(ω) and are therefore at

$$\omega p = + \frac{1}{x} \; dn \left[\left(\frac{2r-1}{2n} \right) K \; , \; \sqrt{1-x^4} \right] \qquad \ldots\ldots (61)$$

The complete transfer function is

$$H(p) = M_0 \cdot \frac{\prod_{s=1}^{n} (\omega - \omega_{Z_S})}{\prod_{r=1}^{n} (\omega - \omega_{p_r})} \qquad \ldots\ldots (62)$$

where ω_{Z_S} is given by (60) and may be complex since

$$dn(x+jy,\; k) = \frac{d.c_1 \, d_1 - jk^2.s.c.s_1}{c_1^2 + k^2.s^2.s_1^2} \qquad \ldots\ldots (63)$$

where

$$\left.\begin{aligned}
c &= cn(x,k) \\
s &= sn(x,k) &&= \sqrt{1-c^2} \\
d &= dn(x,k) &&= \sqrt{1-k^2 s^2} \\
\\
c_1 &= cn(y,\; \sqrt{1-k^2}) \\
s_1 &= sn(y,\; \sqrt{1-k^2}) &&= \sqrt{1-c_1^{\,2}} \\
d_1 &= dn(y,\; \sqrt{1-k^2}) &&= \sqrt{1-(1-k^2)s_1^2}
\end{aligned}\right\} \qquad \ldots\ldots (64)$$

and in this case

$$x = - \left(\frac{1+2s}{2n} \right) K$$

$y = V_0$ where V_0 is given by (58)

$k = \sqrt{1-x^4}$

$\sqrt{1-k^2} = x^2$

A Special Case

The case where the zeros of H(p) lie only on the imaginary frequency axis is of special interest and for this to be true then in (63) $dc_1\, d_1 = 0$.

Now since the function dn cannot be zero for real arguments it follows that $c_1 = cn\,(V_0, x^2) = 0$ therefore $V_0 = K'$ the complete elliptic integral

modulus x^2. ..... (65)

However, from (58)

$$V_0 = \frac{K}{2nK_1}\ sc^{-1}\left[{}^1/t\,\varepsilon,\ \sqrt{1-\varepsilon^4}\right]$$

so that

$$^1/t\,\varepsilon = sc\left[\frac{2nK_1K'}{K},\ \sqrt{1-\varepsilon^4}\right] \qquad ..... (66)$$

Since K,K' and K_1 are all defined by either ε or x then (66) imposes a fixed relation between ε, x,t and n (the order of the function).

This can be reduced by considering the original proposal, illustrated in Fig. 2.3.5 that $F(\omega) = {}^1/F(-\omega)$.

Now $F(\omega) = \varepsilon\, sn\left[\frac{(2nU-K)K_1}{K},\ \varepsilon^2\right]$ and $\omega = -\frac{1}{x}\ dn\left[U,\ \sqrt{1-x^4}\right]$

therefore $\omega = +\frac{1}{x}\ dn\left[U + 2j\,K',\ \sqrt{1-x^4}\right]$

so that a shift in U of 2jK' negates ω.

$$\text{therefore } F(-\omega) = \varepsilon\, sn\left[\frac{(2nU-K)}{K}K_1 + 4j\,\frac{K'K_1}{K}n\ ,\ \varepsilon^2\right]$$

$$= \frac{1}{\varepsilon}\ ns\left[\frac{(2nU-K)}{K}K_1\ ,\ \varepsilon^2\right]$$

$$= {}^1/F(\omega)$$

This is true if $\frac{4K'K_1 n}{K} = K_1'$ ..... (67)

which is true if t = 1 when in (66) $^1/t\,\varepsilon = sc\left[\frac{2nK_1K'}{K}\ \sqrt{1-\ ^4}\right]$
$= sc\left[\frac{K_1'}{2},\ \sqrt{1-\varepsilon^4}\right] = {}^1/\varepsilon$

Fixing t = 1 reduces the relationship between ε, x and n to (from 67)

$$4n\ ^{K'}/K = {}^{K_1'}/K_1$$

Now the elliptic nome 'q' is defined by

$$q = e^{-\pi K'/K} \text{ and therefore}$$

$$q_1 = e^{-\Pi K_1'/K_1} = e^{-4n\Pi K'/K} = q^{4n} \qquad ..... (68)$$

For a given K of modulus k the value of q can be obtained from the series given by Grossman (E7)

$$q = E + 2E^5 + 15E^9 + 150E^{13} + ... \qquad ..... (69)$$

$$\text{where } 2E = \frac{1-\sqrt{k'}}{1+\sqrt{k'}}, \quad k' = \sqrt{1-k^2}$$

In this case if we wish to evaluate the value of x for a given ε and n then q_1 must be evaluated first and in this case $k = \varepsilon^2$.

Then q can be computed from (68) ie. $q = q_1^{1/4n}$ and the inverse approximation of (69) can then be used which is

$$k = 4\sqrt{q}\left[\frac{1+q^2+q^6+...}{1+2_q+2_q^4+...}\right]^2 \qquad ..... (70)$$

and then $\sqrt{1-x^4} = k$ so that x can be found.

In figure 2.3.7 the attenuation available over a given bandwidth is plotted from n = 1 to n = 18. It can be observed that for a given order of function there is a minimum attenuation that can be achieved. Similar curves for quadrature modulation networks have been given by Saraga (C6), Weaver (C8) and Bedrosian (C10).

Poles and Zeros of H(p)

In this case $V_0 = K'$ so that (60) reduces to

$$\omega_Z = -j\,\frac{1}{x}\;cs\left[\left(\frac{1+2s}{2n}\right).K\;,\quad \sqrt{1-x^4}\right] \qquad \ldots\ldots\;(71)$$

and the expression (61) for the poles remains unchanged at

$$\omega_p = +\,\frac{1}{x}\;dn\left[\left(\frac{2r-1}{2n}\right)K_1\;,\quad \sqrt{1-x^4}\right]$$

$$\text{and } H(p) = \prod_{r=1}^{n} \frac{\left(\omega.x + j\;cs\left[\left(\frac{1+2r}{2n}\right)\;K,\;\sqrt{1-x^4}\right]\right)}{\left(\omega.x - dn\;\left[\left(\frac{2r-1}{2n}\right)\;K,\;1-x^4\right]\right)} \qquad \ldots\ldots\;(72)$$

This function has been tabulated using a computer programme is given in Appendix I for n = 2 to n = 9 and for stopband attenuations from 35 dB to 70 dB in steps of 5 dB.

The Relation Between Passband and Stopband Loss

In this special case since t = 1 the passband ripple a_p is related to the stopband attenuation a_s by (from (41) and (42))

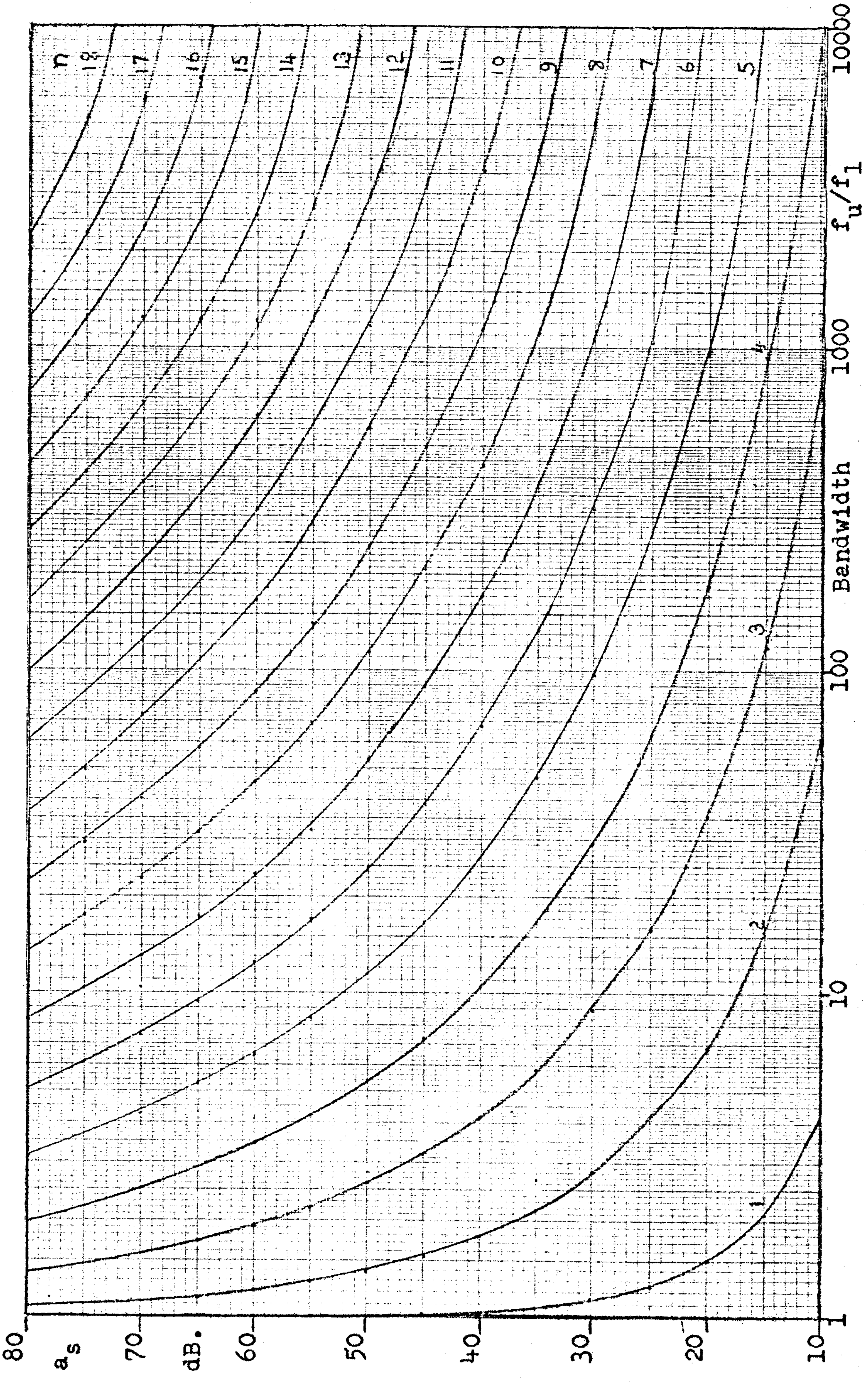


Figure 2.3.7 Elliptic special case. Relation between No. of sections, Attenuation and Bandwidth.

$$a_p = 10 \log_{10} (1 + \varepsilon^2) \text{ dB}$$

$$a_s = 10 \log_{10} (1 + {}^1/\varepsilon^2) \text{ dB}$$

therefore $\varepsilon^2 = (10^{a_s/10} - 1)^{-1}$ ..... (73)

therefore $a_p = 10 \log_{10} \left[1 + (10^{a_s/10} - 1)^{-1}\right]$ ..... (74)

The table below shows the variation of a_p against a_s and it can be seen that for even moderate values of a_s the pass-band ripple is extremely low.

It can also be observed that a_p decreases by approximately a factor of 10 for every 10 db increase in a_s which can be expressed as $a_p \approx \dfrac{4.5}{10^{a_s/10}}$ dB.

Stopband Loss a_s dB	Passband ripple a_p dB
10	0.4576
15	0.1396
20	0.0437
25	0.0138
30	0.0044
35	0.0014
40	0.0004
45	0.00014
50	0.00004

Equivalence Between the Special Case and Known Synthesis Techniques for Phase Splitting Networks

It should be noted that the special case described from p.67-71 gives rise to exactly the same final expression (71) for the zeros of H(p) as is given for the poles and zeros of 90 degree phase splitting networks by a number of authors (reference C6, C8 and C10). The tables given in Appendix I are therefore dual purpose and may be used for designing 90 degree phase splitting networks by considering only the tabulated zeros, ignoring the poles, and forming two functions:

$$H_1(p) = \prod \frac{(p+j\omega_r)}{(p-j\omega_r)} \qquad r = 1,3,5.....$$

and $$H_2(p) = \prod \frac{(p+j\omega_r)}{(p-j\omega_r)} \qquad r = 2,4,6.....$$

The resultant phase splitting networks will be such that when used in a quadrature modulation system the minimum theoretical sideband discrimination achieved will be the attenuation figure given.

2.4 Summary of Synthesis Methods

A wide range of methods exist for designing filters with frequency responses that are asymmetric about zero frequency. Designs may be achieved either from first principles or through the transformation of existing tabulated symmetrical about zero filters. Either transfer function or a complete filter structure can be produced with any of the methods described.

Image filter designs have been studied because of their usefulness in coming to an understanding of the many possible network structures. Further advantages existing in their practical application will be discussed in the next chapter.

Two different direct function synthesis techniques have been studied. The first method is generally applicable to any band pass filter with arbitary stopband requirements. The second method involving elliptic functions has a special case, the solution of which is identical to that of the known problem of designing a 90 degree phase splitting network for single sideband modulation.